Coquelicot

Sylvie Boldo Catherine Lelay Guillaume Melquiond

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Motivations

- Real analysis:
  - partial differential equations,
  - parametric integrals.
- Standard library:
  - dependent types for derivatives and integrals,
  - no support for multivariate functions,
  - lack of uniformity.

Example: Dependent Types

Example

$f(x) = (ax + b)e^{-x}$ is a solution of $f'' + 2f' + f = 0$.

```
Lemma pr1 : derivable f. (* 14 lines of proof *)

Lemma pr2 : derivable (derive f). (* 22 lines *)

Goal ∀ x, derive (derive f pr1) pr2 x
            + 2 * derive f pr1 x + f x = 0.
```

39 lines of Coq just for stating the goal.

Objectives

- Compatibility with standard library.
- User-friendliness.
- Automations.
- Conservative extension.

State of the Art

- ACL2(r): first-order, non-standard analysis.
- Mizar: comprehensive, no automation, set theory.
- PVS: TCC, scope close to Coq.
- HOL Light: comprehensive, net-based topology.
- Isabelle/HOL: comprehensive, filter-based topology.
- Coq/MathClasses: constructive mathematics.

Axioms of Coq' Standard Library

- $\mathbb{R}$: abstract set.
- 0, 1, $+$, $-$, $\times$, $\square^{-1}$, $<$: abstract constants.
- Axioms relating these constants.
- Archimedean: $\texttt{up} : \mathbb{R} \to \mathbb{Z}$.
- Completeness:
  $\texttt{sup} : \forall E : \mathbb{R} \to \texttt{Prop},\ \texttt{bounded}\ E \to \texttt{not\_empty}\ E \to \mathbb{R}$.
- Decidable total order.

Limited Principle of Omniscience

$$\forall P : \mathbb{N} \to \texttt{bool},\ \{n \mid P\ n\} + \{\forall n,\ \neg(P\ n)\}.$$

- Provable with $\mathbb{R}$ axioms.
- Usable to decide `bounded`
  and thus compute `sup` for any nonempty set.

Local Properties

Definition (Local property)

$$\texttt{locally}(x, P) \Leftrightarrow (\exists \delta > 0,\ \forall u \in \mathbb{R},\ |u - x| < \delta \Rightarrow P(u)).$$

Definition (Filter-based continuity)

$$\forall P,\ \texttt{locally}(f(x), P) \Rightarrow \texttt{locally}(x, P \circ f).$$

Definition (Other filters)

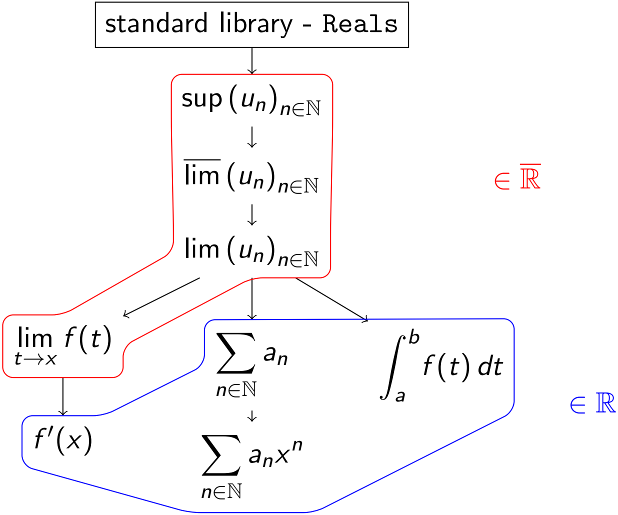
$$\begin{aligned}
\texttt{Rbar_locally}(-\infty, P) &\Leftrightarrow (\exists M \in \mathbb{R},\ \forall u \in \mathbb{R},\ u < M \Rightarrow P(u)) \\
\texttt{eventually}(P) &\Leftrightarrow (\exists N,\ \forall n \in \mathbb{N},\ N \le n \Rightarrow P(n)) \\
\texttt{at_left}(x, P) &\Leftrightarrow \texttt{locally}(x, (u \mapsto u < x \Rightarrow P(u)))
\end{aligned}$$

Type classes: `Filter`, `TopologicalSpace`, `MetricSpace`.

Avoiding Dependent Types

$$\begin{aligned}
\texttt{LimSup_seq}\ u &= \inf(n \mapsto \sup(m \mapsto u_{n+m})) \\
\texttt{Lim_seq}\ u &= \frac{\texttt{LimSup_seq}\ u + \texttt{LimInf_seq}\ u}{2} \\
\texttt{Lim}\ f\ x &= \texttt{Lim_seq}\ \left(n \mapsto f\left(x + \frac{1}{n+1}\right)\right) \\
\texttt{Derive}\ f\ x &= \texttt{Lim}\ \left(h \mapsto \frac{f(x+h) - f(x)}{h}\right)\ 0
\end{aligned}$$

Total Functions



Basic Properties

- Equivalence with definitions from standard library.
- Compatibility with arithmetic operations.
- Rewriting rules.
- And so on.

Rewriting Rules

```
Lemma Derive_scal_r (f : R → R) (k x : R) :
  Derive (fun x ⇒ f x * k) x = Derive f x * k.

Lemma RInt_comp_lin (f : R → R) (u v a b : R) :
  RInt (fun y ⇒ u * f (u * y + v)) a b =
  RInt f (u * a + v) (u * b + v).

Lemma PSeries_incr_n (a : nat → R) (n : nat) (x : R) :
  PSeries (PS_incr_n a n) x = x^n * PSeries a x.
```

Sequences of Functions

Theorem

For any sequence of functions $(f_n)_{n\in\mathbb{N}}$ *and any open set* $D \subset \mathbb{R}$*,*
if $(f_n)_{n\in\mathbb{N}}$ *is uniformly convergent, and*
if $\forall x \in D,\ \forall n \in \mathbb{N},\ \lim_{t\to x} f_n(t)$ *exists, then*

$$\forall x \in D, \quad \lim_{t\to x}\left(\lim_{n\to\infty} f_n(t)\right) = \lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right).$$

Partial Derivatives

Theorem (Schwarz)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

```
Lemma Schwarz : ∀ f x y,
  locally_2d (fun u v ⇒
    ex_derive (fun z ⇒ f z v) u ∧
    ex_derive (fun z ⇒ f u z) v ∧
    ex_derive (fun z ⇒ Derive (fun t ⇒ f z t) v) u ∧
    ex_derive (fun z ⇒ Derive (fun t ⇒ f t z) u) v) x y →
  continuity_2d_pt (fun u v ⇒ Derive (fun z ⇒ Derive (fun t ⇒ f z t) v) u) x y →
  continuity_2d_pt (fun u v ⇒ Derive (fun z ⇒ Derive (fun t ⇒ f t z) u) v) x y →
  Derive (fun z ⇒ Derive (fun t ⇒ f z t) y) x =
  Derive (fun z ⇒ Derive (fun t ⇒ f t z) x) y.
```

Partial Derivatives

Theorem (Taylor-Lagrange approximation)

Under some hypotheses on the bivariate function f,

$$f(x',y') = \texttt{DL\_pol}(f,n,x,y,x'-x,y'-y) + O(\|(x',y')-(x,y)\|^{n+1})$$

with $\texttt{DL\_pol}(f,n,x,y,u,v) =$

$$\sum_{p=0}^{n} \frac{1}{p!} \left(\sum_{m=0}^{p} \binom{p}{m} \cdot \frac{\partial^p f}{\partial x^m \partial y^{p-m}}(x,y) \cdot u^m \cdot v^{p-m} \right).$$

Generalized Limits

Limit points and limit values can be infinite.

Theorem (Compatibility of addition)

```
Lemma is_lim_plus (f g: R → R) (x lf lg:Rbar):
  is_lim f x lf → is_lim g x lg →
  ex_Rbar_plus lf lg →
  is_lim (fun y ⇒ f y + g y) x (Rbar_plus lf lg).
```

Theorem (Intermediate value theorem)

```
Lemma IVT_Rbar_incr (f: R → R) (a b la lb:Rbar) (y:R):
  is_lim f a la → is_lim f b lb →
  (∀ x:R, Rbar_lt a x → Rbar_lt x b →
          continuity_pt f x) →
  Rbar_lt a b → Rbar_lt la y ∧ Rbar_lt y lb →
  {x : R | Rbar_lt a x ∧ Rbar_lt x b ∧ f x = y}.
```

Asymptotic Behaviors

Definition ($f = O(g)$)

```
Definition is_domin (F:(T→Prop)→Prop) (f g: T → R) :=
  ∀ eps:posreal,
  F (fun x ⇒ Rabs (g x) ≤ eps * Rabs (f x)).
```

Definition ($f = O(g)$)

```
Definition is_equiv (F:(T→Prop)→Prop) (f g: T → R) :=
  is_domin F g (fun x ⇒ g x - f x).
```

Theorem ($f \sim g \Rightarrow \lim f = \lim g$)

```
Lemma filterlim_equiv : ∀ (f g : T → R) (l : Rbar),
  is_equiv F f g →
  filterlim f F (Rbar_locally l) →
  filterlim g F (Rbar_locally l).
```

Automation

Example (Differentiation proof)

```
Goal is_derive
  (fun t ⇒ f t + RInt (fun u ⇒ cos (t + u)) 0 1) x l.
Proof
auto_derive.
```

Goals left to prove:

- f is differentiable at x,
- $\int_0^1 \cos(x' + u)du$ exists for any x' in some neighborhood of x,
- $(x', u) \mapsto \sin(x' + u)$ is continuous at any point of $\{x\} \times [0, 1]$,
- $f'(x) + \int_0^1 -\sin(x + u)du = l$.

Baccalaureate S, Mathematics, 2013

Exercises:

1. $f(x) = \dfrac{a + b\ln(x)}{x}$,
2. $u_0 = 2 \quad \text{and} \quad u_{n+1} = \dfrac{2}{3}u_n + \dfrac{1}{3}n + 1 \quad \text{and} \quad T_n = \dfrac{\sum_{k=0}^{n} u_k}{n^2}$.

Notions needed:

- generalized limits for sequences and functions,
- derivatives and variations,
- intermediate values, integrals,
- algebraic manipulations.

Bessel Functions

Definition

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{p=0}^{+\infty} \frac{(-1)^p}{p!(n+p)!} \left(\frac{x}{2}\right)^{2p}.$$

Theorem (Convergence radius)

```
Lemma CV_Bessel1 : ∀ n:nat,
  CV_radius (Bessel1_seq n) = p_infty.
```

Bessel Functions

Theorem (Relations)

$$\begin{aligned}
\forall x \in \mathbb{R}^*, &\quad J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x),\\
\forall x \in \mathbb{R}^*, &\quad J_{n+1}(x) = \frac{nJ_n(x)}{x} - J_n'(x),\\
\forall x \in \mathbb{R}, &\quad J_{n+1}(x) - J_{n-1}(x) = -2J_n'(x).
\end{aligned}$$

Theorem (Differential equation)

```
Lemma Bessel1_correct : ∀ (n : nat) (x : R),
  x^2 * Derive_n (Bessel1 n) 2 x
    + x * Derive (Bessel1 n) x
    + (x^2 - (INR n)^2) * Bessel1 n x = 0.
```

D'Alembert's Formula

Example (1D wave equation)

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t).$$

Solution

$$u(x,t) = \underbrace{\frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right)}_{\alpha(x,t)} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\,d\xi}_{\beta(x,t)}$$

$$+ \underbrace{\frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau}_{\gamma(x,t)}$$

Conclusion

Coquelicot: `http://coquelicot.saclay.inria.fr/`

- Total functions, easy writing of formulas.
- Compatibility with standard library.
- Features: filters, generalized limits, multivariate analysis, automation.

Comprehensive library

- 200 definitions,
- 1,000 lemmas,
- 20,000 lines of Coq,
- no additional axioms.